

On the Classical Zariski Topology Over Prime Spectrum of a Module

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Abstract

Let R be an associative ring with identity and $\text{Spec}(M)$ denote the set of all prime submodules of a right R -module M . In this paper, we study the classical Zariski topology on $\text{Spec}(M)$ which is denoted by τ^c . We prove that if $(\text{Spec}(M), \tau^c)$ is a Noetherian topological space, then M has finitely many minimal prime submodules. We characterize all the irreducible components of $(\text{Spec}(M), \tau^c)$ and all the minimal prime submodules of M for a non-zero flat module M over a commutative ring R . We obtain some results concerning compactness and connectedness of $(\text{Spec}(M), \tau^c)$ by using algebraic properties of the module M . We give some equivalent conditions for $(\text{Spec}(M), \tau^c)$ to be a Hausdorff space or T_1 -space when M is a right module over a left perfect ring R .

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1 Introduction

Throughout this paper all rings will be associative rings with identity elements and all modules will be unital right modules. Unless otherwise stated R will denote a ring. By a proper submodule N of a non-zero right R -module M , we mean a submodule N with $N \neq M$. Given a right R -module M , we shall denote the annihilator of M (in R) by $\text{ann}_R(M)$.

A non-zero R -module M is called a *prime module* if $\text{ann}_R(M) = \text{ann}_R(K)$ for every non-zero submodule K of M . A proper submodule N of a module M is called a *prime submodule* of M if M/N is a prime module. The set of all prime submodules of a module M is called the *prime spectrum* of M and denoted by $\text{Spec}(M)$.

Recall that the spectrum $\text{Spec}(R)$ of a ring R consists of all prime ideals of R . For every ideal I of R , we set $V(I) = \{p \in \text{Spec}(R) : I \subseteq p\}$. Then the sets $V(I)$ satisfy the axioms for the closed sets of a topology on $\text{Spec}(R)$, called the *Zariski topology* (see [11], [12]). In the literature, there are many different generalizations of the Zariski topology of rings to modules. Let M be a right R -module. The *Zariski topology on $\text{Spec}(M)$* is the topology described by taking the set $\Omega = \{V(N) : N \text{ is a submodule of } M\}$ as the set of closed subsets of $\text{Spec}(M)$, where $V(N) = \{P \in \text{Spec}(M) : (P : M) \supseteq (N : M)\}$ (see [6], [8]). The *quasi-Zariski topology on $\text{Spec}(M)$* is described as follows: put $V^*(N) = \{P \in \text{Spec}(M) : P \supseteq N\}$ and $\Omega^* = \{V^*(N) : N \leq M\}$. Then there exists a topology τ^* on $\text{Spec}(M)$ having Ω^* as the family of closed subsets of $\text{Spec}(M)$ if and only if Ω^* is closed under finite unions. When this is the case, τ^* is called the quasi-Zariski topology on $\text{Spec}(M)$ and M is called a top R -module (see [10]). Put $\Omega^c = \{\cap_{i \in I} (\cup_{j=1}^{n_i} V^*(N_{ij})) : N_{ij} \leq M, n_i \in \mathbb{N}, I \text{ is an index set}\}$. Then the elements of Ω^c satisfy the axioms for closed subsets on a topological space. This topology is called the *classical Zariski topology of M* (see [3], [4]). In this paper, we study on the classical Zariski topology of a given module M . In section 2, we give some preliminary results about the classical Zariski topology which will be used in section 3. In section 3, we present our main results for the classical Zariski topology. We prove that there is a bijective correspondence between the set of all minimal prime submodules of a module M and the set of irreducible components of $\text{Spec}(M)$ (see Theorem 3.1). We prove that every minimal prime submodule of a non-zero flat module M over a commutative ring R is of the form Mp for some minimal prime ideal p of $\text{ann}_R(M)$ (see Corollary 3.4). We give an interrelation between compactness of $\text{Spec}(M)$ and finiteness of $\text{Max}(M)$ for a right R -module M (see Theorem 3.5). Then we deal with ultra connectedness and connectedness of $\text{Spec}(M)$ for a right R -module M . We prove that a coatomic right R -module M is hollow if and only if $\text{Spec}(M)$ is ultraconnected (see Theorem 3.6). In Theorem 3.8, we give an interrelation between connectedness of $\text{Spec}(M)$ and decomposition of M for a right R -module M . We give some equivalent conditions for $\text{Spec}(M)$ to be a Hausdorff space or T_1 -space when M is a right module over a left perfect ring R (see Theorem 3.12).

2 Preliminaries

In this section we give some preliminary results about the classical Zariski topology which will be used in Section 3. From now on, for a right R -module M , we consider $\text{Spec}(M)$ with the classical Zariski topology unless otherwise stated.

A topological space X is called a T_1 -space if whenever x and y are distinct points in X , there is a neighborhood of each not containing the other. A topological space X is a T_1 -space if and only if each singleton set in X is closed.

Theorem 2.1 [3, Theorem 2.14] *Let M be a right R -module. Then $\text{Spec}(M)$ is a T_1 -space if and only if every element of $\text{Spec}(M)$ is maximal.*

Proposition 2.2 [3, Proposition 2.20] *Let M be a semisimple right R -module. If $\text{Spec}(M)$ is a T_1 -space, then M is a direct sum of non-isomorphic simple modules.*

A topological space X is called a *Hausdorff space* if, for any two distinct points x, y of X , there exists a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$.

The *cofinite topology* (sometimes called the *finite complement topology*) is a topology which can be defined on every set X . It has precisely the empty set and all cofinite subsets of X as open sets.

Proposition 2.3 [3, Corollary 2.27] *Let M be a right R -module such that $\text{Spec}(M)$ is finite. Then the following statements are equivalent:*

- (1) $\text{Spec}(M)$ is a Hausdorff space.
- (2) $\text{Spec}(M)$ is a T_1 -space.
- (3) $\text{Spec}(M)$ is the cofinite topology.
- (4) $\text{Spec}(M)$ is discrete.
- (5) Either $\text{Spec}(M) = \emptyset$ or every element of $\text{Spec}(M)$ is maximal.

A topological space X is called *irreducible* if $X \neq \emptyset$ and every finite intersection of non-empty open sets of X is non-empty. A (non-empty) subset Y of a topological space X is called an *irreducible subset* if the subspace Y of X is irreducible. For this to be so, it is necessary and sufficient that, for every pair of sets Y_1, Y_2 which are closed in X and satisfy $Y \subseteq Y_1 \cup Y_2$, then $Y \subseteq Y_1$ or $Y \subseteq Y_2$. A maximal irreducible subspace of X is called an *irreducible component*. An irreducible component of a topological space is necessarily closed. Every irreducible subset of X is contained in an irreducible component of X , whence X is the union of its irreducible components.

Lemma 2.4 [3, Lemma 3.3] *Let M be a right R -module. Then for each $P \in \text{Spec}(M)$, $V^*(P)$ is irreducible.*

Let M be a right R -module and $Y \subseteq \text{Spec}(M)$. We denote the intersection of all elements in Y by $I(Y)$.

Theorem 2.5 [3, Theorem 3.4] *Let M be a right R -module and $Y \subseteq \text{Spec}(M)$. If Y is irreducible, then $I(Y)$ is a prime submodule of M .*

For a right R -module M , the prime radical of M (denoted by $\text{rad}(M)$) is defined to be the intersection of all prime submodules of M (note that, if M has no any prime submodule, then $\text{rad}(M) := M$).

Proposition 2.6 [4, Proposition 1.4] *Let M be a right R -module. Then $\text{Spec}(M)$ is homeomorphic to $\text{Spec}(M/\text{rad}(M))$.*

Lemma 2.7 [4, Lemma 1.5] *Let M be a right R -module and $P \in \text{Spec}(M)$. Let $V^*(P)$ be endowed with the induced topology of $\text{Spec}(M)$. Then $V^*(P)$ is homeomorphic to $\text{Spec}(M/P)$.*

3 Main Results

Theorem 3.1 *Let M be a right R -module such that $\text{Spec}(M) \neq \emptyset$. Then there is a bijective correspondence between the set of all minimal prime submodules of M and the set of irreducible components of $\text{Spec}(M)$.*

Proof Let $\text{Min}^p(M)$ denote the set of all minimal prime submodules of M and let $IC(\text{Spec}(M))$ denote the set of all irreducible components of $\text{Spec}(M)$. Let $P \in \text{Min}^p(M)$. Then $V^*(P)$ is an irreducible closed subset of $\text{Spec}(M)$ by Lemma 2.4. Suppose that A is an irreducible subset of $\text{Spec}(M)$ such that $V^*(P) \subseteq A$. Then $I(A) \subseteq P$, and $I(A)$ is a prime submodule of M by Theorem 2.5. By the minimality of P , we have $I(A) = P$. This implies that $V^*(P) = A$ and this shows that $V^*(P)$ is an irreducible component of $\text{Spec}(M)$. Thus we can define the map

$$\psi : \text{Min}^p(M) \longrightarrow IC(\text{Spec}(M)) \text{ by } \psi(P) = V^*(P) \text{ for every } P \in \text{Min}^p(M).$$

Clearly, ψ is well-defined and one to one. Now we show that ψ is surjective. Let $Y \in IC(\text{Spec}(M))$. Then $I(Y)$ is a prime submodule of M by Theorem 2.5. So there is a minimal prime submodule Q of M such that $Q \subseteq I(Y)$. It follows that $V^*(I(Y)) \subseteq V^*(Q)$ and so $Y \subseteq V^*(Q)$. Since $V^*(Q)$ is irreducible by Lemma 2.4, the maximality of Y implies that $Y = V^*(Q) = \psi(Q)$. Thus ψ is a bijective map. \square

A topological space X is said to be *noetherian* if it satisfies the descending chain condition for closed subsets. It is well-known that a noetherian topological space X has a finite number of irreducible components. By using this fact and Theorem 3.1, we obtain the next corollary. The following result generalizes [1, Proposition 3.17] and [4, Propositions 1.6 and 1.8].

Corollary 3.2 *Let M be a right R -module. If $\text{Spec}(M)$ is a Noetherian topological space then M has only a finite number of minimal prime submodules.*

In the following theorem we characterize all the irreducible components of $\text{Spec}(M)$ for a non-zero flat module M over a commutative ring.

Theorem 3.3 *Let R be a commutative ring and M be a non-zero flat R -module. Then every irreducible component of $\text{Spec}(M)$ is of the form $V^*(Mq)$ for some minimal prime ideal q of $\text{ann}_R(M)$. If p is a minimal prime ideal of $\text{ann}_R(M)$ such that $Mp \neq M$ then $V^*(Mp)$ is an irreducible component of $\text{Spec}(M)$.*

Proof Let Y be an irreducible component of $\text{Spec}(M)$. Then $I(Y)$ is a prime submodule of M by [3, Theorem 3.4] and hence $(I(Y) : M) := p$ is a prime ideal of R containing $\text{ann}_R(M)$. Let q be a minimal prime ideal of $\text{ann}_R(M)$ such that $q \subseteq p$. Then $Mq \subseteq Mp \neq M$. Since M is flat Mq is a prime submodule of M by [7, Theorem 3]. By [3, Lemma 3.3], $V^*(Mq)$ is an irreducible closed subset of $\text{Spec}(M)$. Since Y is an irreducible component we have $Y = V^*(Mp) = V^*(Mq)$.

For the last assertion suppose that p is a minimal prime ideal of $\text{ann}_R(M)$ such that $Mp \neq M$. As we mentioned in the proof of the first assertion Mp is a prime submodule of M and $V^*(Mp)$ is an irreducible closed subset of $\text{Spec}(M)$. Let Y' be an irreducible component of $\text{Spec}(M)$ such that $V^*(Mp) \subseteq Y'$. By the first assertion, $Y' = V^*(Mq)$ for some minimal prime ideal q of $\text{ann}_R(M)$. Since Mp is a prime submodule and $V^*(Mp) \subseteq V^*(Mq)$ we get that $q \subseteq p$. By the minimality of p , we have $q = p$. and hence $Y' = V^*(Mp)$. Thus $V^*(Mp)$ is an irreducible component of $\text{Spec}(M)$. \square

Combining Theorem 3.1 and Theorem 3.3, we get the following corollary.

Corollary 3.4 *Let R be a commutative ring and M be a non-zero flat R -module such that $\text{Spec}(M) \neq \emptyset$. Then every minimal prime submodule of M is of the form Mp for some minimal prime ideal p of $\text{ann}_R(M)$.*

Recall that a right R -module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M .

A proper submodule L of a right R -module M is said to be *strongly irreducible* if for any submodules L_1, L_2 of M , $L_1 \cap L_2 \subseteq L$ implies $L_1 \subseteq L$ or $L_2 \subseteq L$.

Theorem 3.5 *Let M be a coatomic right R -module such that every maximal submodule of M is strongly irreducible. The following are satisfied.*

- (1) *If $\text{Max}(M)$ is countable then $\text{Spec}(M)$ is countably compact.*
- (2) *If $\text{Max}(M)$ is finite then $\text{Spec}(M)$ is compact.*

Proof We prove only part (1) since part (2) can be proved similarly. Assume that $\text{Max}(M) = \{N_{\lambda_k}\}$ is countable. Let $\{A_\alpha\}_{\alpha \in I}$ be an open cover of $\text{Spec}(M)$, i.e. $\text{Spec}(M) = \cup_{\alpha \in I} A_\alpha$. Since $\text{Max}(M) \subseteq \text{Spec}(M)$, for each $k \geq 1$, there exists $\alpha_k \in I$ such that $N_{\lambda_k} \in A_{\alpha_k}$. Write $A_{\alpha_k} = \cup_{i_{\alpha_k} \in I_{\alpha_k}} (\cap_{j_{\alpha_k}=1}^{n_{i_{\alpha_k}}} W(L_{i_{\alpha_k}j_{\alpha_k}}))$ for some index set I_{α_k} , $n_{i_{\alpha_k}} (i_{\alpha_k} \in I_{\alpha_k})$ and $L_{i_{\alpha_k}j_{\alpha_k}} \leq M$. For each $k \geq 1$, $N_{\lambda_k} \in \cap_{j_{\alpha_k}=1}^{n_{i_{\alpha_k}}} W^s(L_{i_{\alpha_k}j_{\alpha_k}})$ for some $i_{\alpha_k} \in I_{\alpha_k}$. Suppose that $\sum_{k \geq 1} (\cap_{j_{\alpha_k}=1}^{n_{i_{\alpha_k}}} L_{i_{\alpha_k}j_{\alpha_k}}) \neq M$. By the hypothesis there exists a maximal submodule N_{λ_t} of M for some $t \geq 1$ such that $\sum_{k \geq 1} (\cap_{j_{\alpha_k}=1}^{n_{i_{\alpha_k}}} L_{i_{\alpha_k}j_{\alpha_k}}) \subseteq N_{\lambda_t}$. It follows that $\cap_{j_{\alpha_t}=1}^{n_{i_{\alpha_t}}} L_{i_{\alpha_t}j_{\alpha_t}} \subseteq N_{\lambda_t}$. Since N_{λ_t} is strongly irreducible, $L_{i_{\alpha_t}j_{\alpha_t}} \subseteq N_{\lambda_t}$ for some j_{α_t} ($1 \leq j_{\alpha_t} \leq n_{i_{\alpha_t}}$), a contradiction. Hence $\sum_{k \geq 1} (\cap_{j_{\alpha_k}=1}^{n_{i_{\alpha_k}}} L_{i_{\alpha_k}j_{\alpha_k}}) = M$. It follows that $\text{Spec}(M) = \cup_{k \geq 1} W(\cap_{j_{\alpha_k}=1}^{n_{i_{\alpha_k}}} L_{i_{\alpha_k}j_{\alpha_k}}) \subseteq \cup_{k \geq 1} \cap_{j_{\alpha_k}=1}^{n_{i_{\alpha_k}}} W(L_{i_{\alpha_k}j_{\alpha_k}}) \subseteq \cup_{k \geq 1} A_{\alpha_k}$. This shows that $\{A_{\alpha_k}\}_{k \geq 1}$ is a countable subcover of $\{A_\alpha\}_{\alpha \in I}$. \square

Theorem 3.6 *Let M be a coatomic right R -module. Then M is hollow if and only if $\text{Spec}(M)$ is ultraconnected.*

Proof Assume that M is hollow. Let A, B be two non-empty closed subsets of $\text{Spec}(M)$. Then $A = \cap_{i \in I} (\cup_{j=1}^{n_i} V^*(N_{ij}))$ and $B = \cap_{k \in K} (\cup_{l=1}^{t_k} V^*(L_{kl}))$ for some index sets I, K , n_i ($i \in I$), t_k ($k \in K$) and $N_{ij}, L_{kl} \leq M$. Let $P_1 \in A$ and $P_2 \in B$. Then $P_1 \in V^*(N_{ij})$ for some j ($1 \leq j \leq n_i$) for all $i \in I$ and $P_2 \in V^*(L_{kl})$ for some l ($1 \leq l \leq t_k$) for all $k \in K$. Since M is hollow $P_1 + P_2 \neq M$. By the hypothesis, there exists a maximal submodule P of M such that $P_1 + P_2 \subseteq P$. It can be easily seen that $P \in A \cap B$ whence $A \cap B \neq \emptyset$. Thus $\text{Spec}(M)$ is ultraconnected.

Conversely assume that $\text{Spec}(M)$ is ultraconnected. Let L_1 and L_2 be two proper submodules of M . Then $V^*(L_1) \neq \emptyset$ and $V^*(L_2) \neq \emptyset$ by the hypothesis. By the assumption, $V^*(L_1) \cap V^*(L_2) = V^*(L_1 + L_2) \neq \emptyset$. This implies that $L_1 + L_2 \neq M$. Thus M is hollow. \square

Let p be a prime ideal of R and M be a right R -module. The set of all p -prime submodules of M will be denoted by $\text{Spec}_p(M)$.

Lemma 3.7 *Let M be a right R -module such that every element of $\text{Spec}(M)$ is maximal. Then $|\text{Spec}_p(M)| \leq 1$ for every prime ideal p of R .*

Proof Let p be a prime ideal of R . If $\text{Spec}_p(M) = \emptyset$ then we are done. Suppose that N_1 and N_2 are two p -prime submodules of M . Then $N_1 \cap N_2$ is also a p -prime submodule of M by [9, Proposition 1.10]. By the hypothesis $N_1 \cap N_2 = N_1 = N_2$. This shows that $|\text{Spec}_p(M)| \leq 1$. \square

Theorem 3.8 *Let M be a right R -module. Then the following statements are satisfied.*

(1) *Suppose that M is a coatomic R -module such that $\text{Spec}(M) = \text{Max}(M)$. If $\text{Spec}(M)$ is connected, then M is an indecomposable module.*

(2) *Suppose that R is a right perfect ring and M is a projective right R -module. If M is an indecomposable module, then $\text{Spec}(M)$ is connected.*

Proof (1) Suppose on the contrary that $M = A \oplus B$ for some non-zero submodules A, B of M . Since M is coatomic, $V^*(A)$ and $V^*(B)$ are non-empty closed subsets of $\text{Spec}(M)$. We claim that $\text{Spec}(M) = V^*(A) \cup V^*(B)$. Let P be a p -prime submodule of M . If $A \subseteq P$ or $B \subseteq P$ then $P \in V^*(A) \cup V^*(B)$ and we are done. Suppose that $A \not\subseteq P$ and $B \not\subseteq P$. Then $B + P = M$. We $M/(A \oplus (B \cap P)) = (A \oplus B)/(A \oplus (B \cap P)) \simeq B/(B \cap P) \simeq (B + P)/P = M/P$. This shows that $A \oplus (B \cap P)$ is a p -prime submodule of M . M contains only one p -prime submodule by Lemma 3.7. Thus $A \oplus (B \cap P) = P$ and hence $A \subseteq P$, a contradiction. Thus $\text{Spec}(M) = V^*(A) \cup V^*(B)$. On the other hand $V^*(A) \cap V^*(B) = V^*(A + B) = V^*(M) = \emptyset$. This shows that $\text{Spec}(M)$ is disconnected, again a contradiction. Therefore M is an indecomposable module.

(2) The assertion follows from Theorem 3.6 and the fact that every indecomposable projective module over a right perfect ring is hollow. \square

In the following example we show that the condition on the right R -module M to be coatomic in Theorem 3.8-(1) cannot be removed.

Example 3.9 *Let $R = \mathbb{Z}$ and $M := \mathbb{Z}_q \oplus \mathbb{Z}_{p^\infty}$, where p, q are prime numbers. Then M is not a coatomic module. $\text{Spec}(M) = \text{Max}(M) = \{\mathbb{Z}_{p^\infty}\}$ is connected but M is a decomposable module.*

Let M be an R -module. A non-empty family of submodules N_i ($i \in I$) of a module M is called *coindependent* provided for each $j \in I$ and finite subset J of $I \setminus \{j\}$, $N_j + \bigcap_{i \in J} N_i = M$ and is called *completely coindependent* in case $N_j + \bigcap_{i \neq j} N_i = M$.

The intersection of all maximal submodules of a module M (the Jacobson radical of M) will be denoted by $J(M)$.

Following [13], M is said to have the *max-property* if the maximal submodules of M form a coindependent set of submodules of M and M is said to have the *complete max-property* if the maximal submodules of M form a completely coindependent set. M is said to have the *min-property* provided the simple submodules of M are independent. M is said to have the *direct sum property* provided every maximal submodule of $M/J(M)$ is a direct summand.

Theorem 3.10 *Let M be a right R -module.*

(1) *If M satisfies the direct sum property and $\text{Spec}(M) = \text{Max}(M)$, then $\text{Spec}(M)$ is the discrete space.*

(2) *Let $M/J(M)$ be a semisimple right R -module. Then, $\text{Spec}(M)$ is a T_1 -space if and only if $\text{Spec}(M)$ is the discrete space.*

Proof (1) If $\text{Spec}(M) = \emptyset$, then we are done. Suppose that P_1, P_2 be two maximal submodules of M such that $M/P_1 \simeq M/P_2$. Then $\text{ann}_R(M/P_1) = \text{ann}_R(M/P_2) := p$ and $P_1 \cap P_2$ is a p -prime submodule of M . Since $\text{Spec}(M) = \text{Max}(M)$, we have $P_1 = P_2$. By [13, Theorem 5.5] and [13, Corollary 6.9], M has the complete max-property. This implies that for every maximal submodule L of M , $\{L\} = \text{Spec}(M) \setminus V^*\left(\bigcap_{K \in \text{Max}(M) \setminus \{L\}} K\right)$, an open subset. Since every singleton subset is open, $\text{Spec}(M)$ is the discrete space.

(2) Suppose that $\text{Spec}(M)$ is a T_1 -space. We show that $\text{Spec}(M) = \text{Max}(M)$. To see this, let P be a prime submodule of M . Since $M/J(M)$ is coatomic, there exists a maximal submodule N of M such that $(P + J(M))/J(M) \subseteq N/J(M)$. By Theorem 2.1, we have $P = N$. Thus $\text{Spec}(M) = \text{Max}(M)$. Put $M' := M/J(M)$. By Proposition 2.6, $\text{Spec}(M)$ is homeomorphic to $\text{Spec}(M')$. Thus $\text{Spec}(M')$ is a T_1 -space. Since M' is a coatomic R -module, by Theorem 2.1, $\text{Spec}(M') = \text{Max}(M')$. Thus $\text{Spec}(M')$ is the discrete space by part (1). Now the result follows from Proposition 2.6. The sufficiency is always true. \square

The following result extends [3, Theorem 2.21].

Corollary 3.11 *Let M be a semisimple right module over a PI-ring (ring with a polynomial identity) R . Then the following statements are equivalent.*

- (1) *$\text{Spec}(M)$ is a T_1 -space.*
- (2) *M satisfies the min-property.*
- (3) *$\text{Spec}(M)$ is the discrete space.*

Proof (1) \iff (2) By [3, Theorem 2.21] and [13, Theorem 2.3].

(1) \iff (3) By Theorem 3.10. \square

Theorem 3.12 *Let R be a left perfect ring and M be a right R -module. Then the following statements are equivalent.*

- (1) *$\text{Spec}(M)$ is a Hausdorff space.*
- (2) *$\text{Spec}(M)$ is a T_1 -space.*
- (3) *$\text{Spec}(M)$ is the cofinite topology.*

(4) $\text{Spec}(M)$ is discrete.

(5) Either $\text{Spec}(M) = \emptyset$ or $\text{Spec}(M) = \text{Max}(M)$.

Proof (1) \implies (2) Clear.

(2) \implies (5) By [3, Theorem 2.14], either $\text{Spec}(M) = \emptyset$ or every prime submodule of M is a maximal prime submodule. We may assume that $\text{Spec}(M) \neq \emptyset$. Let P be a prime submodule of M . Since R is a left perfect ring, every right R -module contains a simple submodule. Therefore, M/P is a prime module which contains a simple submodule. By [5, Lemma 1.3], M is a homogeneous semisimple module. By [4, Lemma 1.5], $\text{Spec}(M/P)$ is homeomorphic to $V(P)$. Since the subspace $V(P)$ is a T_1 -space, so is $\text{Spec}(M/P)$. [3, Proposition 2.20] implies that M/P is a simple module, i.e., P is a maximal submodule of M . Thus $\text{Spec}(M) = \text{Max}(M)$.

(5) \implies (2) By [3, Theorem 2.14].

(5) \implies (3) We may assume that $\text{Spec}(M) = \text{Max}(M)$. By Lemma 3.7, there are no distinct maximal submodules P_i, P_j of M such that $(P_i : M) = (P_j : M)$. In other words, there are no distinct maximal submodules P_i, P_j of M such that $M/P_i \simeq M/P_j$. Since R is a left perfect ring, it is a semilocal ring. Therefore there are only finitely many maximal submodules $(P_i)_{i \in I}$ of M such that $M/P_i \not\simeq M/P_j$. Thus $\text{Spec}(M)$ is a finite set. The result follows from [3, Corollary 2.27].

(3) \implies (4) Since every cofinite topology satisfies T_1 -axiom. The proofs of (2) \implies (5) and (5) \implies (3) show that $\text{Spec}(M)$ is a finite set. The result follows from [3, Corollary 2.27].

(4) \implies (5) Since every discrete space satisfies T_1 -axiom, the result follows from the proof of implication (2) \implies (5).

(5) \implies (1) The proof of implication (5) \implies (3) shows that $\text{Spec}(M)$ is a finite set. By [3, Theorem 2.14], $\text{Spec}(M)$ is a T_1 -space. The result follows from [3, Corollary 2.27]. \square

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